Section 8.8 suggests some interesting uses for quadratic equations.

Parabolas EITHER have a maximum value OR a minimum value. If the coefficient of the x-squared term is positive, the parabola opens up and has a minimum and if the coefficient is negative, the parabola opens down and has a maximum value.

The point where this extreme value occurs is called the vertex of the parabola and can be found by completing the square thus:

$$f(x) = -2x^2 + 10x - 7$$

$$f(x) = -2\left(x^2 - 5x + \quad\right) - 7$$ Notice the negative in front of the 5.

$$f(x) = -2\left(x^2 - 5x + \left(\frac{5}{2}\right)^2\right) - 7 + 2\left(\frac{25}{4}\right)$$

$$f(x) = -2\left(x - \frac{5}{2}\right)^2 - \frac{14}{2} + \frac{25}{2}$$

$$f(x) = -2\left(x - \frac{5}{2}\right)^2 + \frac{11}{2}$$

The graphing formula for a parabola is $f(x) = a(x-h)^2 + k$ where (h, k) is the vertex. In our example, that vertex is at $\left(\frac{5}{2}, \frac{11}{2}\right)$.

Another method to get this information is to use the formula for the vertex: $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$.

We have $a = -2$, $b = 10$, $c = -7$ so we get

$$\left(\frac{-10}{2(-2)}, -7 - \frac{10^2}{4(-2)}\right)$$

$$\left(\frac{-10}{-4}, -7 - \frac{100}{-8}\right)$$

$$\left(\frac{5}{2}, -7 + \frac{25}{2}\right)$$

$$\left(\frac{5}{2}, \frac{-14 + 25}{2}\right)$$

$$\left(\frac{5}{2}, \frac{11}{2}\right)$$

Let us consider an application problem where this will be of use.

Example:

A shuttle service operating between two major shopping malls charges \$10 per person and carries 300 passengers per day. The manager estimates that he will lose 15 passengers for each increase of one dollar in the fare. What fare would be most profitable for the shuttle service?

Let x = the number of \$1 increases in fare
$x + 10$ = new price to charge
$300 - 15x$ = new number of passengers.

The function R(x) will represent the revenue earned by the shuttle company.

$R(x) = (300 - 15x)(x + 10)$ The number of items (new passengers) times value each (new fare) = revenue.

$R(x) = 3000 + 150x - 15x^2$ We have $a = -15$, $b = 150$, $c = 3000$ so $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$ gives us $\left(\frac{-150}{-30}, \ .....\right)$

The x-value of the vertex is 5. The x value in our function is the number of \$1 price increases so we know the maximum value will occur when we have 5 one-dollar increases. Thus the new price should be \$15.

If the question had asked, "What is most profit the service can earn?" we would need the R(x) part of the vertex. The y-value.

$\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right)$ would yield: $\left(.... \ , \ 3000 - \frac{150^2}{4(-15)}\right)$ ……. $\left(.... \ , \ 3000 - \frac{150 \cdot 150}{4(-15)}\right)$

Simplifying to get $\left(.... \ , \ 3000 + \frac{75 \cdot 10}{2}\right)$ .... $(.... \ , \ 3000 + 75 \cdot 5)$.......... $(.... \ , \ 3000 + 375)$

So the maximum Revenue is \$3375. This is the range value of the function – the y-value of the vertex.

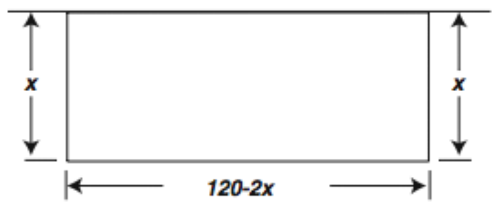
It should not come as a surprise that you must read carefully and then answer the question asked.

Homework problems:

1. In a 110-volt circuit having a resistance of 11 ohms, the power W in watts when a current I is flowing is given by the formula: $W = 110I - 11I^2$. Determine the maximum power that can be delivered in this circuit.

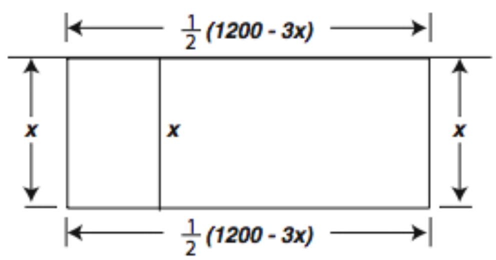
2. A real-estate operator estimates that the monthly profit p in dollars from a building f floors high is given by the formula: $p = -2f^2 + 88f$. How many floors should the real-estate operator consider to be most profitable?

3. A rectangular plot is to be enclosed by using part of an existing fence as one side and 120 feet of fencing for the other three sides. What is the greatest area that can be enclosed? See diagram:



4. Re-work problem 3 assuming that only 90 feet (instead of 120 feet) of fencing is available.

5. A rectangular field is to be enclosed by a fence and divided into two smaller rectangular fields by another fence. Find the dimensions of the field of greatest area which can be thus enclosed and partitioned with 1,200 feet of fencing. See diagram:



6. What is the area of the largest rectangular garden that a farmer can enclose with 500 feet of fencing if one side of the garden will be the side of a barn?

7. A manufacturer has 100 tons of metal that he can sell now with a profit of $5 a ton. For each week that he delays shipment, he can produce another 10 tons of metal. However, for each week he waits, the profit drops 25 cents a ton. If he can sell the metal at any time, when is the best time to sell so that his profit is maximized?

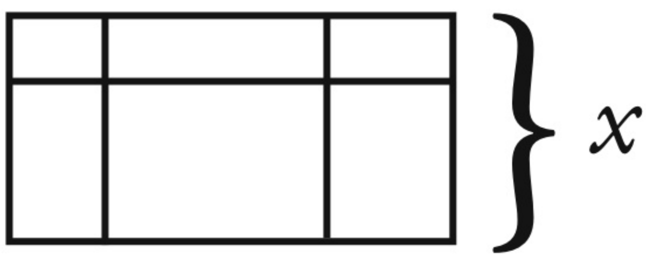
Use: { Let x = weeks to wait
= new # of tons
= new profit per ton

8. A bus company carries 2000 passengers per day at a rate of $2.00 per passenger. In determining a rate hike, it was determined that the company would lose 200 passengers for every 50 cents they increase the rates. What rate should they charge to maximize their income?

Use:
- Let x = # of 50 cent increases
- = new # of passengers
- = new price per passenger

9. A trough with a rectangular cross section is to be made from a long sheet of metal 24 meters wide by turning up strips along each side. Find the amount that must be turned up to give the cross section with greatest area.

10. There are 1200 feet of fencing available to enclose a main rectangular plot of ground that is divided into six unequal rectangles. See the picture. Notice that there are 4 vertical fences and 3 horizontal fences. What are the dimensions of the main rectangle that encloses the maximum area?



Answers to homework problems:

1. The maximum power that can be delivered is 275 watts.
2. The real estate operator should consider 22 floors to earn the most profit.
3. The greatest area enclosed by the fence is 1,800 sq feet.
4. The greatest area enclosed by the fence is 1,012.5 sq feet.
5. The dimensions that will yield maximum area is 200 by 300.
6. The greatest area enclosed would be 8,750 sq feet.
7. The manufacturer should wait 5 weeks before selling.
8. The bus company should charge $3.50 per person.
9. The sheet of metal should be turned up 6 meters from both edges.
10. The original rectangle dimensions should be 150 by 200 feet.

While the final answers are as stated above, to earn credit for the problem, you must have a proper set up and show your ability to use algebra to solve the problem.

Section 9.1 Composite and Inverse Functions

The text gives us a function: $g(x) = 2x + 24$ where the domain, x, represents the shoe size in the U.S. The range, $g(x)$, is the shoe size in Italy.

U.S. size 8 women's is the same as 16+24=40. That is size 40 in Italian shoe size.

The text gives another function: $f(x) = \frac{1}{2}x - 14$ where the domain represents the shoe size in Italy and the range, $f(x)$, represents the shoe size in Britain.

So a size 40 in Italy would be size 6 in Britain.

$$\begin{aligned} f(40) &= \frac{1}{2}(40) - 14 \\ &= 20 - 14 \\ &= 6 \end{aligned}$$

Notice that we started with a size 8 in the U.S. and ended up with size 6 in Britain.

We will define a function $h(x) = f(g(x))$

Where $h(x)$ is defined to be British shoe size and the domain, x, is U.S. shoe size.

$$h(x) = f(g(x))$$
$$h(x) = f(2x + 24)$$
$$h(x) = \frac{1}{2}(2x + 24) - 14$$
$$h(x) = x + 12 - 14$$
$$h(x) = x - 2$$

To convert from U.S. to British shoe sizes, subtract 2 from the U.S. shoe size.

What we have done is called a "composition of functions". $h(x) = f(g(x))$ This is sometime written as $h(x) = (f \circ g)(x)$. While this second version of writing the same thing is a "fancy" to write it, it really doesn't point out that we are, in fact, putting $g(x)$ in for all the x values in $f(x)$. I prefer to use this version: $h(x) = f(g(x))$.

Again, using the function supplied by the text: $g(x) = 2x + 24$ where the domain (x value) is U.S. size and the Range , $g(x)$, is Italian shoe size. This means, given the U.S. size, we would double it and add 24 to get the size in Italy.

If we had the Italian shoe size, can we get the U.S. size?

Thus… size 30 in Italy = ???? size in U.S.

$$30 \overset{set}{=} 2x + 24$$
$$6 = 2x$$
$$3 = x$$ so U.S. size is 3 when the Italian size is 30.

To compute the function that undoes the original we do the following:

$g(x) = 2x + 24$	For ease of manipulation, change $g(x)$ to z.
$z = 2x + 24$	rewrite this line but swap x and z
$x = 2z + 24$	Solve *this* equation for z.
$x = 24 = 2z$	
$\frac{1}{2}x - 12 = z$	Change the z into $g^{-1}(x)$
$g^{-1}(x) = \frac{1}{2}x - 12$	g-inverse function

The $g^{-1}(x)$ is translated into English as "g inverse". It is the inverse function of $g(x)$. What ever $g(x)$ does to a number, $g^{-1}(x)$ undoes.

So, continuing with the story….

Given size 30 from Italy….

$$f^{-1}(x) = \frac{1}{2}x - 12$$
$$f^{-1}(30) = \frac{1}{2}(30) - 12$$
$$f^{-1}(30) = 15 - 12$$
$$f^{-1}(30) = 3$$ so an Italian size 30 shoe is a U.S. size 3

For this inverse function, the domain is the range of the original, ie Italian shoe size and the range is the domain of the original ie. U.S. shoe size.

The first way is easiest if we *only* wanted to find one size change. If we needed to compute many size changes, we would want to use the inverse function.

Many people are confused by the notation $f^{-1}(x)$.

$f(x)$ refers to the *name of a function*. It does **not** mean *f* times *x*. Thus, $f^{-1}(x)$ does **not** mean $\dfrac{1}{f(x)}$. When the "– 1" is on a number or a variable, it is called an exponent such as $3^{-1} = \dfrac{1}{3}$, $a^{-1} = \dfrac{1}{a}$. However, when the "– 1" is on the *name* of a function it means "the inverse of …"

"Whatever a function does, its inverse undoes *and* whatever an inverse does, a function undoes".
This is translated into algebraic notation as: $f^{-1}\left(f(x)\right) = x = f\left(f^{-1}(x)\right)$

More practice with composition….

$$f(x) = \sqrt{x} \qquad g(x) = x - 1$$

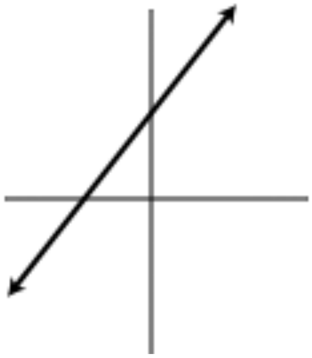
$$h(x)) = f\left(g(x)\right) \qquad j(x) = g\left(f(x)\right)$$
$$h(x)) = f(x-1) \qquad j(x) = g\left(\sqrt{x}\right)$$
$$h(x)) = \sqrt{x-1} \qquad j(x) = \sqrt{x} - 1$$

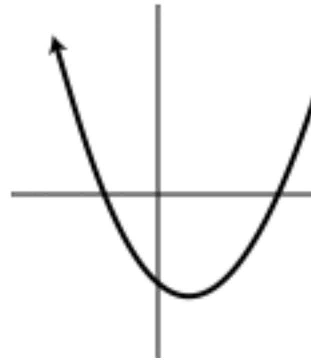
Some functions do not have inverses that are functions.

A function is a relationship that, for each domain item (*x* item) there is **one** range item (*y* item)

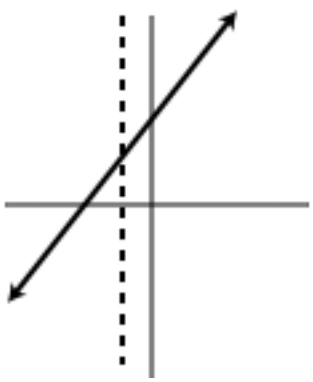
When there is more than one *y* item for a particular domain item, we say we have a "relation".

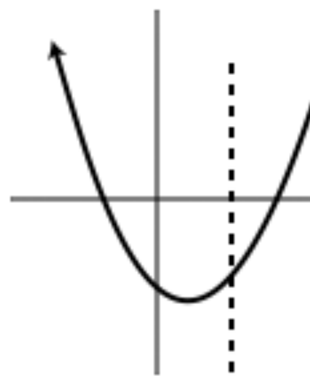
Functions that have an inverse that is also a function are called one-to-one functions. Graphically, the situation is vivid.



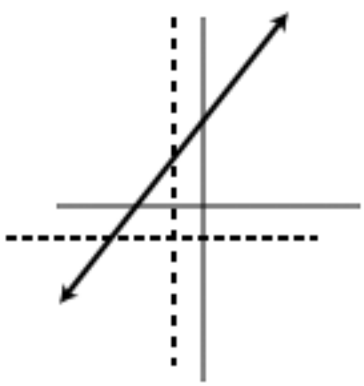


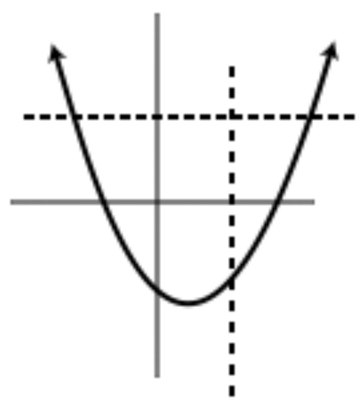
For both, notice that a **vertical line** (which represents a single domain value) crosses the curve only one time. This shows that both are functions.





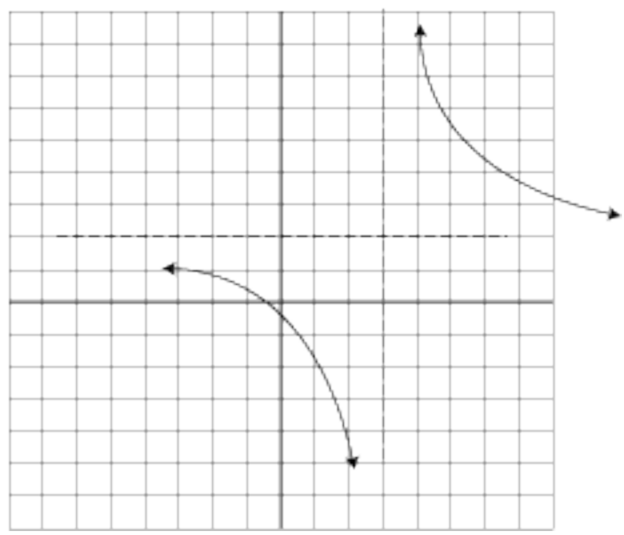
Drawing a horizontal line tells us another story.





Notice that the horizontal line crosses the left function only once but crosses the right hand function **twice**. Thus, while both are functions, the right hand function is not a 1:1 function.

The function $f\left(x\right)=\dfrac{2x+1}{x-3}$ looks like this:



We know that the dotted line at $x = 3$ is a vertical asymptote because $x - 3$ is the denominator of the function. x cannot equal 3 because that would cause division by zero.

Why do we have the dotted line at $y = 2$?

Just by looking at $f\left(x\right)=\dfrac{2x+1}{x-3}$ we can say that $D_f \; x \neq 3$ "by inspection".
The relationship between a function and its inverse is that the roles of the domain and range are exchanged. If we have the inverse of a function, we can determine the domain restrictions on the inverse, which are identical to the range restrictions on the original function.

$$f\left(x\right)=\frac{2x+1}{x-3}$$
$$x=\frac{2z+1}{z-3}$$
$$xz-3x=2z+1$$
$$xz-2z=3x+1$$
$$z\left(x-2\right)=3x+1$$
$$z=\frac{3x+1}{x-2}$$
$$f^{-1}\left(x\right)=z=\frac{3x+1}{x-2}$$

We can, in fact, see (by inspection) that the domain of the inverse does not allow x to be 2 so that is a restriction on the range of our original function.

What is the value of $f(5)$?

$f(5)=\frac{11}{2}$

This means that $f^{-1}\left(\frac{11}{2}\right)=\frac{3\cdot\frac{11}{2}+1}{\frac{11}{2}-2}=\frac{\frac{33}{2}+\frac{2}{2}}{\frac{11}{2}-\frac{4}{2}}=\frac{\frac{35}{2}}{\frac{7}{2}}=\frac{35}{2}\cdot\frac{2}{7}=5$ as we would expect.

$$f\left(f^{-1}(x)\right)=$$

$$f\left(\frac{3x+1}{x-2}\right)=\frac{2\left(\frac{3x+1}{x-2}\right)+1}{\left(\frac{3x+1}{x-2}\right)-3}$$

$$=\frac{\frac{6x+2+x-2}{x-2}}{\frac{3x+1-3(x-2)}{x-2}}$$

$$=\frac{\frac{7x}{x-2}}{\frac{3x+1-3x+6}{x-2}}$$

$$=\frac{7x}{x-2}\cdot\frac{x-2}{7}$$

$$=x$$

$$f^{-1}\left(f(x)\right)=$$

$$f^{-1}\left(\frac{2x+1}{x-3}\right)=\frac{3\left(\frac{2x+1}{x-3}\right)+1}{\left(\frac{2x+1}{x-3}\right)-2}$$

$$=\frac{\frac{6x+3+x-3}{x-3}}{\frac{2x+1-2x+6}{x-3}}$$

$$=\frac{\frac{7x}{x-3}}{\frac{7}{x-3}}$$

$$=\frac{7x}{x-3}\cdot\frac{x-3}{7}$$

$$=x$$

Thus, $f(x)$ undoes what $f^{-1}(x)$ does and $f^{-1}(x)$ undoes what $f(x)$ does!